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Discrete Mathematics 118 (1993) 33–43
North-Holland

Block-transitive t-designs I: point-imprimitive designs

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Received 4 May 1989
Revised 20 December 1991

Abstract

Cameron, P.J. and C.E. Praeger, Block-transitive t-design I: point-imprimitive designs, Discrete Mathematics 118 (1993) 33–43.

We study block-transitive, point-imprimitive t-(v, k, λ) designs for fixed t, v and k. A simple argument shows that we can assume that such a design admits a maximal imprimitive subgroup of S_v. Delandtsheer and Doyen bounded v in terms of k assuming that $t \geqslant 2$; we obtain stronger bounds assuming that $t \geqslant 3$ or that the design is flag-transitive. We also give a structure theorem for designs which attain the Delandtsheer–Doyen bound for all but a few small values of k, and show that for most values of k, there are exactly three such nonisomorphic designs.

1. Preliminaries

A t-(v, k, λ) *design* is a pair $\mathscr{D} = (X, B)$, where X is a set of v points and B a set of k-subsets of X called blocks, such that any t points are contained in exactly λ blocks, where $\lambda > 0$. A design (X, B) is called *trivial* if B consists of all the k-subsets of X. A *flag* in a design is an incident point–block pair. A subgroup G of the automorphism group of $\mathscr{D}$ is *block-transitive* if it acts transitively on B; $\mathscr{D}$ is *block-transitive* if $\mathrm{Aut}(\mathscr{D})$ is. *Point-* and *flag-transitivity* are defined similarly. For more information about t-designs, see [7].

Most of the studies of block-transitive designs hitherto have concerned the case $t = 2$, $\lambda = 1$. By contrast, we consider designs with fixed values of v, k and t but with no restriction on λ. The basis of our method is the following elementary result.

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Proposition 1.1. *Let $\mathscr{D}=(X, B)$ be a t-(v,k,λ) design, admitting a block-transitive group G. Let H be a permutation group with $G \leqslant H \leqslant \mathrm{Sym}(X)$, and $B^* = BH$ the set of images of blocks in B under H. Then (X, B^*) is a t-(v,k,λ^*) design, for some λ^*, admitting the block-transitive automorphism group H. The same is true with 'flag-transitive' replacing 'block-transitive'.*

Proof. It is clear that H acts block-transitively (or flag-transitively) on (X, B^*), and that v and k are unaltered. All that is at issue is that we really have a t-design. However, if we list all the sets bh, where b runs through B and h through a set of coset representatives for G in H, we obtain a t-design, possibly with repeated blocks (the union of $|H:G|$ copies of B), and, by block-transitivity, each block is repeated the same number of times. □

Corollary 1.2. *Suppose that there exists a nontrivial block-transitive t-(v,k,λ) design. Assume that the only k-homogeneous subgroups of S_v are S_v and A_v. Then there exists a nontrivial block-transitive t-(v,k,λ^*) design whose automorphism group is a maximal subgroup of S_v or A_v.*

The assumption in the Corollary, is not very restrictive; it is certainly satisfied if $v \geqslant 2k \geqslant 12$, or if $k=4$ or 5 and $v \geqslant 25$. This follows from the classification of 4-homogeneous groups, a consequence of the classification of finite simple groups (see [1]). We will use this and related results many times.

The point of the corollary, of course, is that it enables us to apply the O'Nan–Scott theorem, which, in its original form (see [12]), gives a classification of the maximal subgroups of S_v or A_v into six types. They are all on the following list (although not all of these are maximal, and it may be necessary to take the intersection with A_v):

(i) $S_m \times S_{v-m}$ in its intransitive action;
(ii) $S_c \wr S_d$ in its imprimitive action, $v=cd$;
(iii) $S_c \wr S_d$ in its product action, $v=c^d$;
(iv) affine groups $\mathrm{AGL}(m,p)$, $v=p^m$, p prime;
(v) diagonal groups $T^d.(\mathrm{Out}(T)\times S_d)$, $v=|T|^{d-1}$, T simple;
(vi) almost simple primitive groups ($T \leqslant G \leqslant \mathrm{Aut}(T)$, T simple).

By Block's Lemma (see [7], Section 1.6), a block-transitive 2-design is point-transitive; so we can ignore type (i). In this paper, we are mainly concerned with describing the designs with automorphism groups of type (ii). Further motivation comes from the fact that, for almost all v, the only primitive groups of degree v are the symmetric and alternating groups (see [2]); so, for such v, any nontrivial block-transitive design is point-imprimitive.

Note that, by the Corollary, if there exists a design admitting a block-transitive, point-imprimitive group, then there is a design (with the same v, k and t) admitting $S_c \wr S_d$ in its imprimitive action. (Recall that this group consists of all permutations of X preserving a partition into d classes each of size c.) So any nonexistence results (phrased just in terms of v, k and t) for designs admitting the wreath product extend immediately to the wider class. In the other direction, we construct some designs admitting the wreath product. While they usually have astronomically large values of λ, the constructions can sometimes be modified, by using a smaller group, to give designs with more practicable parameters.

The connection between groups and designs is also very simple, and is encapsulated in the following folklore result.

Proposition 1.3. *Let G be a permutation group on X, having orbits $O_1, \ldots, O_m$ on the set of t-subsets of X, and b a k-subset of X. Then (X, bG) is a t-design if and only if the ratio of the number of members of O_i contained in b to the total number of members of O_i is independent of i. The group G acts block-transitively on the design, and is flag-transitive if and only if the setwise stabiliser of b in G acts transitively on b.*

The proof is straightforward.

We conclude this section with an example to show that our methods are not confined to imprimitive groups. We intend to return to this topic in a subsequent paper.

Example 1.4. Let G be the symmetric group of degree n, in its action on 2-subsets, with $v = \frac{1}{2}n(n-1)$. Then G has two orbits on 2-subsets, whose sizes are in the ratio 1 to $\frac{1}{4}(n-3)$. A k-set can be regarded as the edge set of a graph (on the vertex set $\{1, \ldots, n\}$) with k edges. We obtain a block-transitive 2-design if and only if the ratio of the number of pairs of edges with a vertex in common to the total number $\frac{1}{2}k(k-1)$ of pairs of edges in $1/\frac{1}{4}(n+1)$. The design is flag-transitive if and only if the graph is edge-transitive.

For a simple example, take the Petersen graph, with $k = 15$ edges and 30 intersecting pairs of edges. We require $\frac{1}{4}(n+1) = 105/30$ or $n = 13$, giving a flag-transitive 2-(78, 15, λ) design, for some (quite large) λ.

2. Block-transitive, point-imprimitive designs

To begin, we observe some natural limitations on designs with the properties of the title.

Proposition 2.1. (i) *A block-transitive automorphism group of a nontrivial* 4*-design is* 2*-homogeneous on points.*

(ii) *A flag-transitive automorphism group of a nontrivial* 3*-design is* 2*-transitive on points.*

Proof. (i) The theorem of Ray-Chaudhuri and Wilson [10] shows that the incidence matrix of 2-sets against blocks has full rank. Now the argument of Block's lemma gives the result.

(ii) The derived design is a block-transitive, and hence point-transitive, 2-design. □

So, in the point-imprimitive case, we need to consider block-transitive 2- and 3-designs, and flag-transitive 2-designs, only. Suppose that we have such a design, whose automorphism group preserves a partition of the points into d classes of size c. By Corollary 1.2, we may assume that its automorphism group is $S_c \wr S_d$. Any block is partitioned by its intersections with the congruence classes: let $x_1, x_2, \ldots,$ be the sizes of these intersections. This partition has at most d parts, each of size at most c. Then the design is completely determined: the blocks are all k-sets partitioned into parts of sizes $x_1, x_2, \ldots,$ by the congruence classes. Denote this structure by $\mathscr{D}(c, d; \boldsymbol{x})$, where $\boldsymbol{x} = (x_1, x_2, \ldots,)$.

Set $b_t = \sum x_i(x_i - 1) \cdots (x_i - t + 1)$. Note that $b_1 = k$.

Proposition 2.2. *Let $\mathscr{D} = \mathscr{D}(c, d; x)$, with $cd = v$ and $b_1 = k$.*

(i) *$\mathscr{D}$ is always a 1-design.*

(ii) *$\mathscr{D}$ is a 2-design if and only if*

$$b_2 = \frac{k(k-1)(c-1)}{(v-1)}.$$

(iii) *$\mathscr{D}$ is a 3-design if and only if it is a 2-design and*

$$b_3 = \frac{k(k-1)(k-2)(c-1)(c-2)}{(v-1)(v-2)}.$$

(iv) *$\mathscr{D}$ is never a nontrivial 4-design.*

Proof. (i) is immediate from the transitivity of $G = S_c \wr S_d$, and (iv) from Proposition 2.1. Now G has two orbits on pairs, and (provided that $c \geqslant 3$ and $d \geqslant 3$) three orbits on triples; so the conditions of Proposition 1.4 for a 2-design and a 3-design involve one and two equalities, respectively. It is readily checked that these reduce to the ones given. If either of c or d is 2, then one of the orbits on triples is empty, but the result is still valid. □

3. Block-transitive 3-designs

For that values of c, d, k can there exist a 3-design on $v = cd$ points with automorphism group transitive on blocks and imprimitive on points, with d congruence classes of size c? Our analysis shows that this is equivalent to the two equations of Proposition 2.2 holding, for some $\boldsymbol{x}$ with $\sum x_i = k$.

From these two equations, we derive

$$b_2(v-1)=k(k-1)(c-1),$$

and

$$b_3(v-2)=b_2(k-2)(c-2),$$

two linear equations for v and c. Since $(v-1)/(v-2)\neq(c-1)/(c-2)$, the equations are independent. Hence the partition x determines c and v (and hence d). The cases $c=2$ and $d=2$ are special; we deal with these first.

If $c=2$, then no part of the partition x exceeds 2. The converse is also true; for, if $c>2$, then (since we have a 3-design) some block contains three points in the same congruence class.

Proposition 3.1. *Let x be the partition $(1^{k-2q}, 2^q)$. Then the following are equivalent:*
(i) *$\mathscr{D}(2, d; x)$ is a 3-design;*
(ii) *$\mathscr{D}(2, d; x)$ is a 2-design;*
(iii) *$2d=1+k(k-1)/2q$.*

Proof. The equivalence of (i) and (ii) is immediate from the paragraph before the statement of the proposition; for we have $c=2$ and $b_3=0$, so the equation of Proposition 2.2(iii) is vacuously true. Now the equivalence of (ii) and (iii) is Proposition 2.2(ii). □

Corollary 3.2. *A block-transitive, point-imprimitive 3-design with $c=2$ satisfies $v\leqslant\frac{1}{2}k(k-1)+1$.*

In a similar way, $d=2$ if and only if the partition x has just two parts.

Proposition 3.3. *Let x be the partition $(\frac{1}{2}(k+u), \frac{1}{2}(k-u))$. Then the following are equivalent:*
(i) *$\mathscr{D}(c, 2; x)$ is a 3-design;*
(ii) *$\mathscr{D}(c, 2; x)$ is a 2-design;*
(iii) *$2c=1+k(k-1)/(k-u^2)$.*

Proof. The equivalence of (ii) and (iii) is Proposition 2.2(ii) again. Then substitution shows that the equation of Proposition 2.2(iii) is satisfied. □

Note that necessarily $u^2<k$ and u has the same parity as k; hence:

Corollary 3.4. *A block-transitive, point-imprimitive 3-design with $d=2$ has $v\leqslant\frac{1}{2}k(k-1)+1$.*

The bounds in the corollaries are best possible; choose $q=1$ in 3.1, $k=u^2+2$ in 3.3.

Conjecture 3.5. Any block-transitive, point-imprimitive 3-design satisfies $v \leqslant \frac{1}{2}k(k-1)+1$.

We have been unable to show this from the equations of Proposition 2.2. However, it is supported by numerical evidence. As noted earlier, c and d are determined by the partition $\boldsymbol{x}$ of k. We have computed all partitions of k with $k \leqslant 70$ for which c and d are integers greater than 2 and satisfy the further condition that the number of parts is at most d while the greatest part is at most c. The conjecture is valid in all these cases, although the conjectured bound is met surprisingly often. The first few values of $k, c, d, \boldsymbol{x}$ are:

$$20, 11, 7, (5, 4^2, 2^3, 1)$$

$$20, 7, 11, (4, 3^3, 2^3, 1^4)$$

$$27, 22, 16, (5, 2^{11})$$

$$27, 22, 16, (4^2, 3^2, 2^3, 1^7)$$

$$27, 16, 22, (4, 3, 2^6, 1^8)$$

$$27, 16, 22, (3^5, 1^{12})$$

$$27, 8, 44, (3, 2^4, 1^{16}).$$

Note that it often happens that two (or more) partitions of the same value of k occur having c and d interchanged. We do not understand this phenomenon, but we found a way to exploit it to construct block-transitive 3-designs with smaller automorphism groups (and much smaller values of λ). In place of the wreath product, we use the direct product $S_c \times S_d$ acting on the product of sets of sizes c and d. The points can be regarded as the cells of a rectangular array or as the edges of the complete bipartite graph $K_{c,d}$. A subset of size k can be regarded as a bipartite graph on $c+d$ vertices with k edges. If Γ is such a graph, let $\mathscr{D}(c, d; \Gamma)$ be the design whose blocks are the images under $S_c \times S_d$ of Γ.

Proposition 3.6. *Let Γ be a bipartite graph with bipartite blocks of sizes c and d, and let $\boldsymbol{x}$ and $\boldsymbol{y}$ be the degree sequences of vertices in the blocks of sizes d and c, respectively.*

(i) *$\mathscr{D}(c, d; \Gamma)$ is a 2-design if and only if $\mathscr{D}(c, d; \boldsymbol{x})$ and $\mathscr{D}(d, c; \boldsymbol{y})$ are 2-designs.*

(ii) *$\mathscr{D}(c, d; \Gamma)$ is a 3-design if and only if*

(a) *$\mathscr{D}(c, d; \boldsymbol{x})$ and $\mathscr{D}(d, c; \boldsymbol{y})$ are 3-designs, and*

(b) *the number of paths of length 3 in Γ is*

$$\frac{k(k-1)(k-2)(c-1)(d-1)}{(v-1)(v-2)},$$

where $v = cd$.

The proof is just an application of Proposition 1.3.

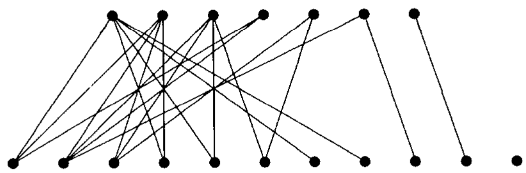

Fig. 1. A 3-design.

Note that the existence of a graph Γ for which the design is a 2-design really depends only on the partitions $\boldsymbol{x}$ and $\boldsymbol{y}$, via the Gale–Ryser theorem on the existence of bipartite graphs with specified vertex degrees ([5, 11]), but for 3-designs, a new condition must be imposed.

In the case $\{c, d\} = \{7, 11\}$, using the partitions given above, we require a bipartite graph with prescribed vertex degrees and having 72 paths of length 3. The first such graph was found by Sheila Cameron, and is shown in Fig. 1; there are others. In these cases, designs with still smaller λ can be obtained by replacing $S_7 \times S_{11}$ by a smaller subgroup, for example $A_7 \times M_{11}$.

4. Flag-transitive 2-designs

Proposition 4.1. *A flag-transitive, point-imprimitive 2-design satisfies* $v \leqslant (k-2)^2$. *This bound is attained for all even* $k > 4$.

Proof. It suffices to consider $\mathscr{D}(c, d; \boldsymbol{x})$, where $\boldsymbol{x}$ is a partition with all parts equal, say q parts of size $p(pq = k;\ p, q > 1)$. Then Proposition 2.2(ii) gives

$$(p-1)(cd-1) = (c-1)(pq-1).$$

From this, we find

$$c\left(\frac{(pq-1)}{(p-1)} - d\right) = \frac{p(q-1)}{(q-1)}.$$

The coefficient of c on the left-hand side is at least $1/(p-1)$; so $c \leqslant p(q-1)$. Now

$$\begin{aligned} v - 1 &= cd - 1, \\ &\leqslant (pq - p - 1)(pq - 1)/(p-1), \\ &= (k - p - 1)(k-1)/(p-1). \end{aligned}$$

The right-hand side is a decreasing function of p; so $v - 1 \leqslant (k-3)(k-1)$, giving the result. $\square$

Equality holds if and only if $p = 2$, $c = d = 2(q-1)$.

There is a remarkable complementary result of Davies [3].

Theorem 4.2. *In a flag-transitive, point-imprimitive* 2-(v, k, λ) *design, v is bounded by a function of λ.*

5. Block-transitive 2-designs

The first result on this subject, which suggested to us the possibility of our approach, was the following theorem of Delandtsheer and Doyen [4].

Theorem 5.1. *A block-transitive, point-imprimitive 2-design satisfies*

$$v \leqslant (\tfrac{1}{2}k(k-1)-1)^2.$$

We will give a proof, because we will be investigating the case of equality, and also because we have all the machinery in place already. Let $K = \frac{1}{2}k(k-1)$, $y = \frac{1}{2}b_2$. By Proposition 2.2(ii),

$$K(c-1) = (cd-1)y.$$

Now $d-1 = cd-1-d(c-1)$; so $cd-1$ divides $K(d-1)$, say

$$K(d-1) = (cd-1)z.$$

Multiplying these equations, and dividing by $v-1$, gives

$$K^2 - K(y+z) = (v-1)yz,$$

so

$$v = \frac{(K-y)(K-z)}{yz}$$

Now $(K-t)/t$ is a decreasing function of t; so $v \leqslant (K-1)^2$.

Equality holds if and only if $y = z = 1$, that is, $c = d = K-1$. The bound is attained by $\mathscr{D}(K-1, K-1; x)$, with $x = (1^{k-2}, 2)$. By Proposition 3.5, it is also attained by $\mathscr{D}(K-1, K-1; \Gamma)$, where Γ is a bipartite graph with k edges and all vertices of degree 1 except for one of degree 2 in each bipartite block. There are just two such graphs; one consists of a path of length 3 and $k-3$ isolated edges, the other two parts of length 2 and $k-4$ isolated edges. (Note that the full automorphism group of both these designs is the primitive group $S_K \wr S_2$, with the product action.)

We will show that, for most values of k, the three designs just described are the only 2-designs which admit block-transitive, point-imprimitive groups and attain the bound of Delandtsheer and Doyen. Our result is more general, describing the possible automorphism groups of such designs for all but a few small values of k in terms of the simple 2-transitive groups of degree $K-1$. Set $m = K-1$.

Theorem 5.2. *Let $\mathscr{D}$ be a 2-(v, k, λ) design, with $v = m^2$, $m = \frac{1}{2}k(k-1) - 1$, $k > 5$, $k \neq 8$. Let G be a group of automorphisms of $\mathscr{D}$ which acts block-transitively and point-imprimitively. Then one of the following happens:*

(i) *G is a subgroup of* $\mathrm{Aut}(T) \wr S_m$ containing T^m, *where T is a simple 2-transitive group of degree m, and G projects onto a 2-transitive subgroup of S_m;*

(ii) *$T_1 \times T_2 \leqslant G \leqslant \mathrm{Aut}(T_1) \times \mathrm{Aut}(T_2)$, where T_1 and T_2 are simple 2-transitive groups of degree m.*

Corollary 5.3. *Let k be such that the only 2-transitive groups of degree $m = \frac{1}{2}k(k-1) - 1$ are S_m and A_m. Then there are just three nonisomorphic 2-(v, k, λ) designs with $v = m^2$ admitting block-transitive, point-imprimitive groups.*

The three designs of the corollary are those described after Theorem 5.1.

Remark. The following are equivalent, for $k > 5$ and $k \neq 8$:

(i) the only 2-transitive groups of degree $m = \frac{1}{2}k(k-1) - 1$ are S_m and A_m;

(ii) $m = \frac{1}{2}k(k-1) - 1$ is not of the form $(q^d - 1)/(q-1)$, for q a prime power and $d \geqslant 2$.

For it follows from the list of 2-transitive groups (see [1]) that the degree m of a 2-transitive group other than S_m or A_m is a prime power, or $(q^d - 1)/(q-1)$ (q a prime power), or $2^{2d-1} \pm 2^{d-1}$ or 22 176 or 276. If $m = \frac{1}{2}(k+1)(k-2) = p^d$ (p prime), then $k+1$ and $k-2$ are of the form p^t or $2p^t$. If $p \nmid k-2$, then $k-2 = 1$ or 2, $k \leqslant 4$. Otherwise, p divides both $k+1$ and $k-2$, and so $p = 3$; and $k-2 = 3$ or 6, $k = 5$ or 8. Furthermore, m cannot be $2^{2d-1} \pm 2^{d-1}$, since these are triangular numbers, while m is one less than a triangular number, and m is not 22, 176 or 276. So the claim is proved.

The equation

$$\tfrac{1}{2}k(k-1) - 1 = (q^d - 1)/(q-1)$$

(q a prime power) has solutions for $d = 2$. There are probably infinitely many solutions of $\frac{1}{2}k(k-1) - 1 = p + 1$, p prime, although this is not known; we found 8075 solutions for $9 \leqslant k \leqslant 65\,536$, with some evidence that they are thinning out. It is interesting to note that, in this range, there are only five solutions of $\frac{1}{2}k(k-1) - 1 = p^t + 1$ with p prime and $t > 1$, and only one (viz. $k = 12$) with $t > 2$.

On the other hand, we *conjecture* that the diophantine equation

$$\tfrac{1}{2}k(k-1) - 1 = (q^d - 1)/(q-1)$$

has no solutions at all with $d \geqslant 3$ (without even restricting q to be a prime power). Certainly there are none with $d = 3$ or with $q = 2$ or with $k \leqslant 65\,536$. Hering [6] has shown that there are only finitely many solutions for any given value of q.

These remarks justify our claim that Corollary 5.3 applies for most values of k.

Moreover, even when the corollary does not apply, Theorem 5.2 provides a strong restriction on $\mathrm{Aut}(\mathscr{D})$, and the blocks of $\mathscr{D}$ form a single orbit under this group, so it is possible in particular cases to determine all such designs.

Corollary 5.4. *There is no block-transitive, point-imprimitive* 2-$(v, k, 1)$ *design with* $v=(\frac{1}{2}k(k-1)-1)^2$, $k\geqslant 9$.

For any block meets one class in two points and a further $k-2$ classes in just one point, see the proof of the theorem. Hence, in case (i) of the theorem, we have $\lambda\geqslant m^{k-2}$, since, given two points P, Q in the same congruence class, if b is one block containing them, then any k-set containing P and Q and meeting the same congruence classes is a block. In case (ii), if P and Q lie in the same row of the square array, their stabiliser acts transitively on the remaining $m-1$ rows, and so at least $(m-1)/(k-2)$ blocks contain P and Q.

The remaining cases in the corollary have been settled by O'Keefe et al. (to appear) and Nickel et al. (submitted). There are no designs at all except in the case $k=8$, where there are over 400 non-isomorphic designs (all of which have been determined)!

Proof of Theorem 5.2. First note that m is not a prime power (by the analysis following Corollary 5.3), and $m\neq 28$. From the list of 2-transitive groups, we deduce that:

(i) a 2-transitive group of degree m has a 2-transitive simple normal subgroup;
(ii) a 2-homogeneous group of degree m is 2-transitive.

Now let $\mathscr{D}$ and G be as in the statement of the theorem. The proof of Theorem 5.1 shows that any congruence on points has m classes of size m, and that any block meets one class in 2 points and $k-2$ further classes in just one point. Let Y be a congruence class, and H the permutation group induced on it by its setwise stabiliser. Then H is transitive on the blocks meeting Y in 2 points, and so is 2-homogeneous (and hence 2-transitive) on Y. Let T be its simple 2-transitive normal subgroup.

Next we show that, dually, G acts 2-homogeneously (and hence 2-transitively) on the set of congruence classes. Let e be the length of an orbit of G on unordered pairs of congruence classes. Then G has a fixed set S on unordered pairs of points of size em^2. If b is the number of blocks, then $\lambda=2b/m^2(m-1)$, and so the number of incidences between pairs in S and blocks is $2be/(m-1)$. By block-transitivity, this number is a multiple of b; so $m-1$ divides $2e$. But e is the length of an orbit on pairs of a transitive group of degree m; so m divides $2e$. We conclude that $m(m-1)$ divides $2e$, and that G is 2-homogeneous on the set of congruence classes, as claimed.

Consider first the case that G acts faithfully on the set of congruence classes. Then G is a 2-transitive group of degree m, in which the stabiliser of a point has a section isomorphic to the simple group T, which itself acts 2-transitively on the m points of Y. Inspection of the 2-transitive groups shows this to be impossible. (Only for $m=6$, 8, 12 does this occur, and none of these is of the form $\frac{1}{2}k(k-1)-1$.)

So the kernel N of the action of G on the congruence classes is nontrivial. Obviously $N^Y\geqslant T$, so the socle of N is a direct power of T. Call two classes 'equivalent' if the same direct factor of the socle acts nontrivially on them. This is a G-congruence; by the 2-transitivity of G, it is a trivial congruence, and so the socle of N is either T^m or T.

If the socle of N is T^m, we are finished.

Now consider the case where the socle is T. Since T has at most two inequivalent permutation representations of degree m (a further consequence of the classification of 2-transitive groups), we can call two classes equivalent if T acts on them in the same way, obtaining at most two 'equivalence' classes, and hence (as above) just one class. This set of m fixed points is a class of another congruence, 'orthogonal' to the original congruence, and the stabiliser of one of these new classes in 2-transitive on it. Each point lies on one class of each family, and G preserves a square grid. The result is proved. □

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